Set Operations

Two sets are called disjoint if their intersection is empty, that is, they share no elements:

$A \cap B = \emptyset$

The difference between two sets A and B contains exactly those elements of A that are not in B:

$A-B = \{x \mid x \in A \wedge x \notin B\}$

Example: $A = \{a, b\}$, $B = \{b, c, d\}$, $A-B = \{a\}$

Cardinality: $|A-B| = |A| - |A \cap B|$

Set Operations

The complement of a set A contains exactly those elements under consideration that are not in A: denoted A^c (or $\overline{A}$ as in the text)

$A^c = U - A$

Example: $U = \mathbf{N}$, $B = \{250, 251, 252, \ldots\}$

$B^c = \{0, 1, 2, \ldots, 248, 249\}$

Logical Equivalence

Equivalence laws

- Identity laws, $P \wedge T \equiv P$,
- Domination laws, $P \wedge F \equiv F$,
- Idempotent laws, $P \wedge P \equiv P$,
- Double negation law, $\neg(\neg P) \equiv P$
- Commutative laws, $P \wedge Q \equiv Q \wedge P$,
- Associative laws, $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$,
- Distributive laws, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$,
- De Morgan's laws, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
- Law with implication $P \rightarrow Q \equiv \neg P \vee Q$

Set Identity

Table 1 in Section 1.7 shows many useful equations

- Identity laws, $A \cup \varnothing = A$, $A \cap U = A$
- Domination laws, $A \cup U = U$, $A \cap \varnothing = \varnothing$
- Idempotent laws, $A \cup A = A$, $A \cap A = A$
- Complementation law, $(A^c)^c = A$
- Commutative laws, $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative laws, $A \cup (B \cup C) = (A \cup B) \cup C$, …
- Distributive laws, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, …
- De Morgan's laws, $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$
- Absorption laws, $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$
- Complement laws, $A \cup A^c = U$, $A \cap A^c = \varnothing$

Set Identity

How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

Method I: logical equivalent

$x \in A \cup (B \cap C)$

$\Leftrightarrow x \in A \lor x \in (B \cap C)$

$\Leftrightarrow x \in A \lor (x \in B \land x \in C)$

$\Leftrightarrow (x \in A \lor x \in B) \land (x \in A \lor x \in C)$ (distributive law)

$\Leftrightarrow x \in (A \cup B) \land x \in (A \cup C)$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Every logical expression can be transformed into an equivalent expression in set theory and vice versa.

Set Operations

Method II: Membership table

1 means "x is an element of this set"

0 means "x is not an element of this set"

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

… and the following mathematical appetizer is about…

Functions

Functions

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write

f(a) = b

if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write

$f: A \rightarrow B$

(note: Here, "$\rightarrow$" has nothing to do with if… then)

Functions

If $f:A \to B$, we say that A is the domain of f and B is the codomain of f.

If $f(a) = b$, we say that b is the image of a and a is the pre-image of b.

The range of $f:A \to B$ is the set of all images of all elements of A.

We say that $f:A \to B$ maps A to B.

Functions

Let us take a look at the function $f: P \to C$ with

P = {Linda, Max, Kathy, Peter}

C = {Boston, New York, Hong Kong, Moscow}

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = New York

Here, the range of f is C.

Functions

Let us re-specify f as follows:

f(Linda) = Moscow
f(Max) = Boston
f(Kathy) = Hong Kong
f(Peter) = Boston

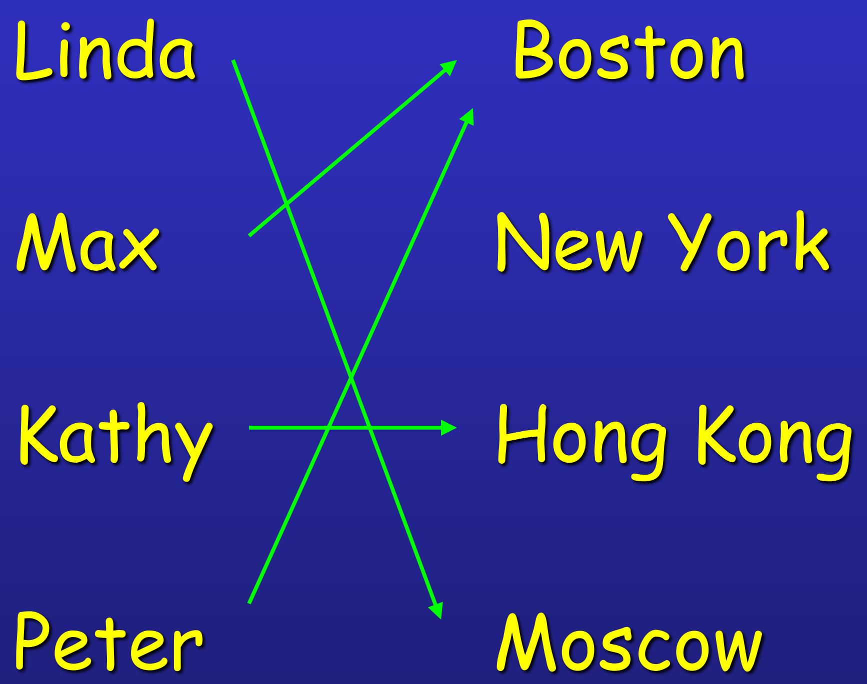
Is f still a function? yes

What is its range? {Moscow, Boston, Hong Kong}

Functions

Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Functions

If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:

$f:\mathbf{R} \rightarrow \mathbf{R}$

$f(x) = 2x$

This leads to:

$f(1) = 2$

$f(3) = 6$

$f(-3) = -6$

…

Functions

Let f_1 and f_2 be functions from A to **R**.

Then the sum and the product of f_1 and f_2 are also functions from A to **R** defined by:

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$(f_1f_2)(x) = f_1(x)\, f_2(x)$

Example:

$f_1(x) = 3x, \; f_2(x) = x + 5$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5$

$(f_1f_2)(x) = f_1(x)\, f_2(x) = 3x\,(x + 5) = 3x^2 + 15x$

Functions

We already know that the range of a function $f: A \to B$ is the set of all images of elements $a \in A$.

If we only regard a subset $S \subseteq A$, the set of all images of elements $s \in S$ is called the image of S.

We denote the image of S by $f(S)$:

$f(S) = \{f(s) \mid s \in S\}$

Functions

Let us look at the following well-known function:

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

What is the image of S = {Linda, Max} ?

f(S) = {Moscow, Boston}

What is the image of S = {Max, Peter} ?

f(S) = {Boston}

Properties of Functions

A function $f:A \to B$ is said to be one-to-one (or injective), if and only if

$\forall x, y \in A \ (f(x) = f(y) \to x = y)$

In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.